Modeling Nonlinear Dynamical Systems with Delay-differential Equations.

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Abstract

We describe a method to model nonlinear dynamical systems using periodic solutions of delay-differential equations. We show that any finite-time trajectory of a nonlinear dynamical system can be loaded approximately into the initial condition of a linear delay-differential system. It is further shown that the initial condition can be extended to a periodic solution of the delay-differential system if an appropriate choice of its parameters is made. As a result, any finite set of trajectories of a nonlinear dynamical system can be modeled with arbitrarily small error via a set of periodic solutions of a linear delay-differential equation. These results can be extended to some non-linear delay differential systems. One application of the method is for modeling memory and perception.

1 Introduction

How the information about the outside world is stored in the brain is still in many ways an open question. Clearly, representation of the totality of the information involves some sort of modeling of the environment by the brain under the specific conditions of its operation, e.g., the limited number of the neurons and synapses, the inherent delays in signal propagation along the neural fibers, *etc*. Assuming that the environment is described by some nonlinear deterministic dynamical system and that the brain is described by another dynamical system, this question can be rephrased as the question of how one dynamical system can model another. Since the environment is typically much more complicated than the brain, the modelling task should be impossible simply on the count of the difference in the number of the degrees of freedom for the two dynamical systems. A possible solution to this puzzle is to assume that the environment is actually a collection of weakly interacting subsystems, each of

which has a lower complexity than the brain. A good strategy to learn complicated environment would then be to model the strongly coupled degrees of freedom group by group. Decomposition of the environment into a collection of components that interact weakly presumably should involve some version of nonlinear factor analysis extended to deterministic systems.

Even if the complexity of the environment is less than that of the system that models it a fundamental question arises about in what sense and how one dynamical system can model another. In addition one can ask whether there exist a class of dynamical systems that in some sense are universal in their modeling properties in the sense that a large class of nonlinear dynamical systems that represent possible environments can be modeled using essentially the same system. Of course, in order for this to make sense in explaining the properties of the brain, the modeling has to be physically realizable.

Leaving investigation of the deterministic nonlinear factor analysis to another publication [1], in this paper we present arguments that when the dimension of the environment is less or equal to that of the modeling system such universal modeling systems exist and that it is by the essential use of the delays that the universality can be realized. We prove that under certain conditions any nonlinear dynamical system can be modeled by a linear delay-differential dynamical system with constant coefficients, in the sense that any finite set of trajectories of the environment can be loaded into periodic solutions of the delay-differential equation. The loading turns out to be equivalent to solving an essentially linear problem and is achieved by adjusting the coefficients of the delay-differential equation in a way that is plausible from the biological point of view. With the biological applications in mind we also extend some of the results to a class of nonlinear systems with delays.

Modeling the environment is usually referred to as construction memory models of the environment. Since the early eighties much work has been done on memory models under the assumption that time-independent memories can be represented as fixed points of systems of nonlinear differential equations (ODE) that possess a Liapunov function and with the evolution given by

$$\dot{x}\,(t) = F\,(x)\,,\ \ x(t) \in R^N \tag{1}$$

for some function $F\,(x)$. Once a Liapunov function is given, the fixed points of the evolution can be identified with the N-dimensional stored patterns. Domains of attraction of the fixed points then can be viewed as all the patterns that will be "associatively" recalled dynamically as the stored pattern represented by the fixed points. Especially useful within this approach proved to be the Hopfield model [2] described as a dynamical system with evolution

$$\dot{x}\,(t) = -E_\mu \cdot x\,(t) + A \cdot \sigma(x\,(t)) + y, \tag{2}$$

where $E_\mu = diag\{\mu_i\},\ \mu_i > 0$ is an $N \times N$ diagonal matrix, A - is an $N \times N$ matrix, called the matrix of weights, $\sigma : x_i \rightarrow \sigma_i(x_i)$ is a component-wise nonlinear transformation and the constant vector $y \in R^N$ describes time-independent external inputs to the system. If $\sigma'(z) \geq 0$ and matrix A is symmetric, or if

the symmetric part of A is non-positive definite [3], then the system (2) has a Liapunov function $V(x)$ defined on the configuration space such that on the trajectories $\dot{V}(x(t)) < 0$, leading to the convergence of all trajectories to a set of fixed points. An extensive theory exists about the properties of memory storage, especially for random patterns [4].

A certain amount work has been done for the systems with delays. Existence of a Liapunov function can be proven for the Hopfield model with delays if the delays are not too large [5]. However, the fixed-points paradigm is difficult to apply to the time dependent patterns. Time-variable patterns, which in our approach are represented by the trajectories of nonlinear dynamical systems, are ubiquitous in the environment. Therefore, understanding the underlying principles of their storage is important when constructing plausible models for perception and memory. Some interesting effects of the presence of delays on the structure of the attractors leading to multistability in networks of two Hodgkin-Huxley type spiking neurons were discussed heuristically and numerically in [6]. From our point of view the model that is considered there is that of a system of two linear delay differential equations with time variable coefficients. In this paper we do not restrict ourselves to the models of spiking neurons but consider arbitrarily large systems of linear and nonlinear delay-differential equations with slowly varying coefficients. Systems with time variable coefficients will be considered in detail elsewhere. Nevertheless, our results are in general agreement with the numerical simulations in [6] and provide a new perspective on their interpretation.

Here we present an approach to the time-dependent pattern storage that describes sets of interacting "neurons" as systems of delay-differential equations (DDE) with multiple delays. Since delays in nerve impulse propagations are common, the approach is more plausible biologically than, for example, the Hopfield model where instantaneous communication among the neurons is assumed. Mathematically the delays appear after integrating out some internal degrees of freedom in systems of Hodgkin-Huxley equations [7]. Although more complicated to analyze, the DDE have some properties that ordinary differential equations do not have. For example, for a solution of a DDE to be unique, one has to specify an initial condition which is a function defined on a finite time range [8]. This is a key feature of DDE that we exploit.

Our starting point is to include explicitly the multiple delays $\tau_1, ..., \tau_L$ in the dynamics of memory model so that instead of (1) we obtain a system of nonlinear DDE

$$\dot{x}(t) = F(x(t), x(t-\tau_1), ..., x(t-\tau_L)). \tag{3}$$

One example of such a system is a generalization of the Hopfield model to include delays. This was considered in [5]

$$\dot{x}(t) = \sum_{k=0}^{L} A_k \cdot \sigma(x(t-\tau_k)) + y(t), \tag{4}$$

where as before the A_k and are $N \times N$ weight matrices with $A_0 = -E_\mu$, and the delays are defined so that $\tau_0 = 0, \tau_k \le \tau_{k+1}$. In contrast to (2) we assume the external inputs $y(t)$ as time-variable. In models describing networks of oscillating neurons one can also consider τ_k as an $N \times N$ matrix with each element $(\tau_k)_{ij}$ describing the delay in signal propagation from neuron (j) to neuron (i) via "k"th pathway. However, after appropriate redefinition of matrices A_k and nonlinearities $\sigma(z)$ one can recover (4) at the expense of increasing L to the number of different values of all matrix elements τ_k of all $(L+1)$ matrices. We assume that $y(t)$ is a trajectory of some nonlinear dynamical system with local evolution given by $\dot{y}(t) = G(y)$. The detailed nature of $G(y)$ is not important for the discussion. When $\sigma_i(z) = z$ we obtain a system of linear DDE. Writing it in components we obtain

$$\dot{x}_i = \sum_{k=0}^{L} (A_k)_{ij} \cdot x_j (t - \tau_k) + y_i(t),\ i = 1, ..., N. \tag{5}$$

The main point of this paper is to prove the existence of the periodic solutions of systems of homogeneous linear DDE that are periodic extensions of their initial conditions. That is, we show that, given a set of fixed delays τ_k, the weight matrices A_k can be chosen in such a way that the initial condition when extended periodically to $t > \tau_L$ is the solution of (5). With appropriate choice of parameters the choice of A_k is unique. We also show that if the inhomogeneous part of a DDE is given by a trajectory of a nonlinear dynamical system, the trajectory can be loaded approximately, but with arbitrarily small error into the initial condition for the DDE and hence stored as its periodic solution. With some modifications these results can be carried over to a class of nonlinear DDE. Using the method we describe, with a sufficiently large N, any finite set of trajectories of a nonlinear dynamical system, for example any finite set of periodic orbits, can be encoded as periodic solutions of a linear DDE. Since many dynamical systems, are uniquely characterized by the sets of their periodic orbits, we can speak of linear modelling of nonlinear systems.

The paper is structured as follows. In the next section we give a brief overview of the differences between the linear ODE and the linear DDE from mathematical point of view. Section 3 contains discussion of loading the trajectories into the initial conditions. In Section 4 we construct solutions for the periodic extension problem. In Section 5 we discuss the adaptation dynamics for the weight matrices A_k. Section 6 is a summary.

2 Delay-differential Equations

To specify a solution of a differential equation uniquely one needs to pose the initial value problem. There is a big difference in how the initial value problem is posed for ODE and for DDE. For the ODE the initial value problem is given by

$x(\phi, y)(t)$ is a solution of (1) for $t > 0$;

$$x(\phi, y)(0) = \phi \in R^N.$$

For the DDE of retarded type (3) with the maximum delay τ_L the initial value problem is formulated as

$$x(\phi, y)(t) \text{ is a solution of (3) for } t > \tau_L;$$

$$x(\phi, y)(0) = \phi(t) \in R^N \text{ for } t \in [0, \tau_L].$$

Hence to specify a unique solution of a DDE one needs an initial value from a space of functions defined on the interval $[0, \tau_L]$. Exploiting the difference is one key feature of our approach.

Let us consider in more detail the difference in the linear case. Similar to homogeneous linear ODE we can search for the exponential solutions of homogeneous DDE. As for ODE these correspond to zeroes of the characteristic equation that is obtained using the Laplace transform. For linear ODE the characteristic equation is polynomial and has exactly N complex roots. The situation is different for linear DDE. The characteristic equation becomes transcendental and is given by

$$\det(H(\lambda)) = 0, \tag{6}$$
$$H(\lambda) = \lambda \cdot I - \sum_{k=0}^{L} A_k \exp(-\lambda \tau_k).$$

There are generically infinite number of complex roots λ_p of (6). Let $N(a, b)$, $a, b \in R$, $a \leq b < \infty$, be the number of the complex roots of (6) in the vertical strip of the complex plain with real part of each point contained in $[a, b]$. Then, some of useful properties of the roots are

(i) $\mathrm{Re}(\lambda_p) \leq c < \infty$ if all $\tau_k \geq 0$, for all k, i, j;

(ii) $\mathrm{Re}(\lambda_p) \to -\infty \qquad p \to \infty$;

(iii) $N(a, b) < \infty$.

Solutions of linear *inhomogeneous* DDE can be conveniently written down in an integral form using the notion of the fundamental solution. The fundamental solution is defined as the unique matrix solution of (5) with $y(t) = 0$ and with the following initial condition

$$X(t) = 0 \quad t \in [0, \tau_L];$$

$$X(t) = I \quad t = \tau_L, \text{ where } I \text{ is the unit matrix.}$$

Using inverse Laplace transform, the fundamental solution can be expressed in terms of the characteristic (matrix) function $H(\lambda)$

$$X(t) = \frac{1}{2\pi i} \int_{(c)} d\lambda \, e^{\lambda t} H^{-1}(\lambda), \tag{7}$$

where (c) is the contour $(c - i\infty, c + i\infty)$ in the complex plane such that $\mathrm{Re}\,(\lambda_p) < c$ and where the Laplace transform and its inverse are defined by

$$\hat{x}\,(\lambda) = \int_0^\infty d\lambda\ \exp\left(-\lambda\ t\right)\ x\,(t)\,, \tag{8}$$

$$x\,(t) = \frac{1}{2\pi i}\int_{(c)} d\lambda\ \exp\left(-\lambda\ t\right)\ \hat{x}\,(\lambda)\,. \tag{9}$$

With these definitions the unique solution of the initial value problem for (5) can be written as [8]

$$\begin{aligned} x\,(\phi, y)\,(t) = X\,(t - \tau_L)\,\phi\,(\tau_L) + \int_{\tau_L}^{t} d\tau\ X\,(t-\tau)\ y\,(\tau) \\ + \sum_{k=1}^{L}\int_{\tau_L - \tau_k}^{\tau_L} d\tau\ X\,(t - \tau - \tau_k)\ A_k\ \phi\,(\tau)\,. \end{aligned} \tag{10}$$

If we put $L = 0$ then the third term in (10) vanishes and we recover the solution of a linear inhomogeneous ODE in terms of its fundamental solution. Note that in (10) the inhomogeneous part $y\,(t)$ and the initial condition $\phi\,(t)$ play a similar role. We shall exploit the similarity below when we generate initial conditions from the inhomogeneous part of the equation.

3 Loading Trajectories into the Initial Conditions

In this section we describe how to generate the initial condition for a DDE from its inhomogeneous part. The main idea is to consider the inhomogeneous part to be nonzero only on $[0, \tau_L]$ and generate the initial condition recursively by adding more and more terms to the equation successfully.

Let us first consider the linear DDE. We divide the interval $[0, \tau_L]$ into adjoining intervals $[\tau_k, \tau_{k+1}]$, $k = 0, ..., L-1$ and for time $t < \tau_m$ truncate the full DDE to

$$\dot{x}\,(t) = \sum_{k=0}^{m} A_k \cdot x\,(t - \tau_k)\,. \tag{11}$$

This can be done under the assumption that the signals with larger delays did not arrive yet at the "neuron" to influence its state.

When $m = 0$ (11) reduces to a linear ODE which can be integrated for $t \in (0, \tau_1]$ using the initial condition ϕ_0 at $t = 0$ and the values of the inhomogeneous part $y(t)$. The result of the integration can be considered as the initial condition the truncated DDE defined on $t \in (\tau_1, \tau_2]$, which enables us to integrate the DDE with one delay $\tau_1$on the segment $[\tau_1, \tau_2]$ using $y(t)$ defined on $[\tau_1, \tau_2]$ only. Proceeding in this fashion step by step we can construct the initial condition on the entire segment $[0, \tau_L]$. Taking the inhomogeneous part $y(t)$ to be zero

outside of $[0,\tau_L]$ we can interpret the result of this procedure as a linear mapping of trajectories into the initial conditions, $I_L : y(t) \rightarrow \Phi\left(t\right), t \in [0,\tau_L]$.

To describe the iteration procedure in more detail we give the initial iteration step followed by the "m"th iteration. For the initial step we need to generate the initial condition for

$$\begin{aligned} &\dot{x}\left(t\right) = A_0 \cdot x\left(t\right) + A_1 \cdot x\left(t-\tau_1\right) + y(t), \\ &y(t) = 0, t \notin \left[0,\tau_1\right]. \end{aligned} \tag{12}$$

These are given by the solution of the initial value problem

$$\begin{aligned} &\dot{x}\left(t\right) = A_0 \cdot x\left(t\right) + y(t), \\ &x(0) = \phi_0. \end{aligned} \tag{13}$$

Using the fundamental solution for this equation the solution for $t > 0$ and the initial condition for (12) can be written as

$$\begin{aligned} &\Phi_0\left(t\right) = X_0\left(t\right) \cdot \phi_0 + \int_0^t ds\ X_0\left(t-s\right)\ y\left(s\right), \\ &X_0\left(t\right) = \exp\left(A_0 t\right) \end{aligned} \tag{14}$$

With this initial condition the solution for (12) can be written in terms of its fundamental solution as

$$\begin{aligned} \Phi_1\left(t\right) = {} & X_1\left(t-\tau_1\right)\ \Phi_0\left(\tau_1\right) - \int_0^{\tau_1} ds\ X_1\left(t-s-\tau_1\right)\ A_1\ \Phi_0\left(s\right) \\ & + \int_{\tau_1}^t ds\ X_1\left(t-s\right)\ y\left(s\right). \end{aligned} \tag{15}$$

$\Phi_1\left(t\right)$ can be considered as the initial condition for the 2nd step of iteration, the result of which is the initial condition defined on the segment $\left[\tau_{1,}\tau_2\right]$. The general form of iteration is easily deduced. Namely,

$$\begin{aligned} \Phi_k\left(t\right) = {} & X_k\left(t-\tau_k\right)\ \Phi_{k-1}\left(\tau_k\right) - \int_{\tau_{k-1}}^{\tau_k} ds\ X_k\left(t-s-\tau_k\right)\ A_k\ \Phi_{k-1}\left(s\right) \\ & + \int_{\tau_k}^t ds\ X_k\left(t-s\right)\ y\left(s\right). \end{aligned} \tag{16}$$

with the final step resulting in the initial condition

$$\Phi_L\left(t\right) = \left\{\Phi_k\left(t\right),\ t \in \left[\tau_k, \tau_{k+1}\right], k = 0, ..., L-1\right\}. \tag{17}$$

Next note that to enable its periodic extension the initial condition has to be periodic, since it has to be defined on a closed segment. Hence we obtain a constraint of the form

$$\Phi_L\left(\tau_L\right) = \Phi_L\left(0\right) \equiv \phi_0. \tag{18}$$

This condition can be used to eliminate an unknown parameter ϕ_0 and make trajectory loading unique. For example, when there is only one delay we obtain

$$\phi_0 = \left(\exp\left(-A_0\tau_L\right) - 1\right)^{-1} \int_0^{\tau_L} ds \exp\left(-A_0 s\right)\; y(s). \tag{19}$$

When multiple delays are present ϕ_0 can be determined similarly. The null space of the mapping I_L is also of interest. It describes all the trajectories that cannot be loaded into the periodic extensions of the initial conditions. For the example with only one delay this space consists of all functions with the Laplace transform satisfying

$$\hat{y}\left(\lambda\right) + \phi_0 = 0 \tag{20}$$

For arbitrary set of delays the null space still forms a one-parameter family and, therefore, its existence excludes only a very small set of all trajectories from loading.

Obviously, the detailed loading procedure described above works only for the linear DDE. However the same principle can be applied to the nonlinear DDE as well. If the delays are known, then one can proceed with the same iterative scheme by numerical integration. We speculate that some sort of the analog integration version of the procedure above might be used in the brain to implement the loading. With this in mind we can interpret the time τ_L as the "attention span" of the population of the neurons that encode a particular time-dependent pattern. For neurons with a large number of synapses and, hence a large number of random delays, a continuous version of the iteration in (16) can be easily written.

4 Periodic Extensions of the Initial Conditions

Having shown how trajectories can be loaded into the initial conditions of the linear DDE, we now construct solutions of the DDE that are the periodic extension of their initial conditions. First we consider the linear case and then discuss the modifications added by the presence of nonlinearities.

4.1 Linear DDEs

Consider a Fourier series representation of a space of periodic and continuous initial conditions on $[0, \tau_L]$

$$\phi\left(t\right) = \sum_{n=-Q}^{Q} \exp\left(i\rho_n t\right) \cdot b_n,\; t \in \left[0, \tau_L\right], \tag{21}$$
$$\rho_n = \frac{2\pi}{\tau_L} n,$$

where b_n are fixed N-dimensional complex-valued vectors such that under complex conjugation $\bar{b}_n = b_{-n}$ (to ensure that $\phi\left(t\right)$ is real-valued). If the initial

condition does not belong to this space we shall consider (21) as an approximation of the initial condition with the error of approximation determined by the truncation of the infinite Fourier series that represents it at the "Q"th term of the expansion.

Let us extend the domain of definition of the initial condition $\phi(t)$ to all $t \geq 0$ by treating $\phi(t)$ as a continuous periodic function for $t \geq 0$ with $\phi(t+\tau_L) = \phi(t)$. Assume now that all delays τ_k are fixed but we are free to vary the matrices A_k. The problem of finding periodic extensions is then to find such A_k that for $t \geq 0$ the function $\phi(t)$ is a solution of the homogeneous equation

$$\dot{\phi}(t) = \sum_{k=0}^{L} A_k \cdot \phi(t-\tau_k). \tag{22}$$

After the substitution of (21) into (22) we obtain a set of linear equations on matrices A_k

$$\left(i\rho_n I - \sum_{k=0}^{L} A_k \cdot \exp(-i\rho_n \tau_k) \right) \cdot b_n = 0,\ \ n = 0, ..., Q. \tag{23}$$

Note that the equations for $n = -Q, ..., -1$ are redundant, since A_k are real and hence the additional equations can be obtained from (23) by complex conjugation. Altogether, because the $n = 0$ equation is real, (23) is equivalent to $(2Q+1)N$ real equations on $(L+1)N^2$ elements of A_k, provided that all elements of A_k are generically non-zero. In the case a delay τ_k is given by a random value of delays in propagation from "i"th to "j" neurons, with probability one only one of the elements of A_k is non-zero (see the discussion in the Introduction). However, in such a case, if $L'+1$ is the number of $N \times N$ delay matrices, then $L+1 = (L'+1)N^2$ and, therefore, the count of unknowns is the same. We conclude that (23) can have a unique solution for A_k only when $(2Q+1) = (L+1)N$. Since Q determines the number of terms in the Fourier expansion and, therefore, the accuracy of representation of an arbitrary initial condition, this relation means that for a given number L of delays and N the number of neurons involved in modeling one can achieve only limited accuracy of representation. Namely, the number of expansion terms Q is bounded so that $(2Q+1) \leq (L+1)N$.

The system (23) can be solved by noticing that it implies that vectors b_n are eigenvectors of linear combinations of A_k with eigenvalues $i\rho_n$. Unlike the typical eigenvalue problem where A_k are known and one needs to find ρ_n and b_n, here the situation is reversed: one needs to find A_k assuming that ρ_n and b_n are known. If an eigenvalue λ and an eigenvector b are given then all the matrices D for which they solve the eigenvalue problem can be written as

$$D = \lambda I + B\left(I - \|b\|^{-2}\, b \otimes \bar{b}\right),$$

where B is arbitrary $N \times N$ matrix and $\|b\|^2 = \sum_{i=1}^{N} |b_i|^2$ and $b \otimes \bar{b}$ is the matrix that is the outer product of b with its complex conjugate. For given eigenvalue

and eigenvector matrix D has $N^2 - N$ free parameters. Therefore, an equivalent way to write (23) is

$$\sum_{k=0}^{L} R_{nk} \cdot A_k = i\rho_n I + B_n \left(I - \|b_n\|^{-2} b_n \otimes \bar{b}_n \right), \; n = 0, ..., Q, \tag{24}$$

$$R_{nk} = \exp\left(-i\rho_n \tau_k\right), \tag{25}$$

where B_n is an $N \times N$ complex matrix, which effectively has $N^2 - N$ free parameters. This linear system of equations always has solutions for appropriate choice of parameters Q, L, N. For some choices the solution is unique. This concludes the proof of existence of the periodic extensions.

Let us consider $(23, 24)$ for various choices of parameters. When $N = 1$ the second term in the RHS of (24) vanishes and the dependence on b_n drops out. Consequently, the system (23) becomes a system of $(2Q+1)$ real linear equations on $L+1$ real parameters A_k

$$\sum_{k=0}^{L} S_{nk} A_k = -\rho_n, \; n = 1, ..., Q, \tag{26}$$

$$\sum_{k=0}^{L} C_{nk} A_k = 0, \qquad n = 0, 1, ..., Q, \tag{27}$$

$$S_{nk} = \sin\left(\rho_n \tau_k\right), \tag{28}$$

$$C_{nk} = \cos\left(\rho_n \tau_k\right). \tag{29}$$

When $L = 2Q$, provided $\det T \neq 0$, the equations can be solved uniquely by inversion of the matrix T

$$T = \begin{pmatrix} \sin\left(\rho_1 \tau_0\right) & ... & \sin\left(\rho_1 \tau_L\right) \\ ... & ... & ... \\ \sin\left(\rho_Q \tau_0\right) & ... & \sin\left(\rho_Q \tau_L\right) \\ 1 & ... & 1 \\ ... & ... & ... \\ \cos\left(\rho_Q \tau_0\right) & ... & \cos\left(\rho_Q \tau_L\right) \end{pmatrix} \tag{30}$$

For arbitrary N, $L = 2Q$, and $\det T \neq 0$ we can write down the solutions for (23) R as

$$A_k = \sum_{k=0}^{2Q} T^{-1}_{kn} \Gamma_n, \qquad k = 0, 1, ..., L. \tag{31}$$

$$\Gamma_n = \left(\rho_n I + \mathrm{Im}\left(B_n \left(I - \|b_n\|^{-2} b_n \otimes \bar{b}_n \right)\right)\right), \quad n = 1, ..., Q \tag{32}$$

$$\Gamma_n = \mathrm{Re}\left(B_n \left(I - \|b_n\|^{-2} b_n \otimes \bar{b}_n \right)\right), \qquad n = 0, 1, ..., Q \tag{33}$$

Since the matrices Γ_n contain $\left(N^2 - N\right)$ free parameters, to obtain a unique solution one needs to increase the value of Q by factor of $\left(N^2 - N\right)$ to constrain the additional degrees of freedom.

Some remarks about the meaning of the periodic extensions should be made. Since the initial conditions were taken as a linear combination of exponential solutions, the existence of the extensions is equivalent to the existence of $2Q+1$ pair-wise conjugate zeroes of the characteristic function that are located on the imaginary axis. As mentioned in the previous section, there can be only a finite number of zeroes of the characteristic function in any finite width vertical strip of the complex plane. Hence, there could be only a finite number of zeros of the characteristic function lying on the imaginary axis. As a result, Q *must* be finite and only finite-dimensional spaces of initial conditions can be extended periodically. A practical consequence of this is that one can store a trajectory as a periodic extension only approximately, with the error of approximation given by the error induced by truncating the Fourier series. However, as follows from the theory of Fourier expansions, the error can be made arbitrarily small at least in L^2 norm by increasing N.

4.1.1 Non-linear DDE

Let us now consider the existence of the periodic extensions in the presence of nonlinearities. Take, for example, the extended Hopfield model with dynamics

$$\dot{x} = \sum_{k=0}^{L} A_k \cdot \sigma\left(x\left(t-\tau_k\right)\right) + y(t) \tag{34}$$

Proceeding as before we expand the periodic the initial condition in the Fourier series for $t \in [0, \tau_L]$ as in (21) and obtain that for $\phi(t)$ to be a solution of homogeneous DDE it needs to satisfy

$$\dot{\phi}\left(t\right) = \sum_{k=0}^{L} A_k \cdot \sigma\left(\phi\left(t-\tau_k\right)\right) \tag{35}$$

Substitution of (21) into (35) and expansion of the nonlinear term in Fourier series yields a system of equations on the unknown matrices A_k

$$\left(i\rho_n b_n - \sum_{k=0}^{L} A_k \cdot \exp\left(-i\rho_n \tau_k\right) \cdot \Lambda_n\left(\tau_k; \{b_m\}\right)\right) = 0,\ n = -Q_1, ..., Q_1 \tag{36}$$

where Q_1 is not necessarily equal to Q and for each $p = 1, ...N$ the coefficient $\left(\Lambda_n\left(\tau_k; \{b_m\}\right)\right)_p$ is defined by

$$\Lambda_n\left(\tau_k; \{b_m\}\right) = \frac{1}{\tau_L}\int_0^{\tau_L} dt \exp\left(-i\rho_n \ \left(t-\tau_k\right)\right)\ \sigma\left(\sum_{m=-Q}^{Q} \exp\left(i\rho_m \ \left(t-\tau_k\right)\right) \cdot b_m\right) \tag{37}$$

The additional time-independent factor $\exp\left(i\rho_n \ \tau_k\right)$ in the definition of $\Lambda_n\left(\tau_k; \{b_m\}\right)$ ensures that if $\sigma\left(z\right) = z$ then $\Lambda_n\left(\tau_k; \{b_m\}\right) = b_n$.

Now the vector coefficients of the Fourier expansion (21) cannot be factored out and the solution for A_k will depend not only on the delays τ_k but also on $b_n$even for $N = 1$. Such dependence is desirable in a memory model, since it implies that the choice of the parameters A_k is pattern specific.

When $Q_1 \neq Q$ and $Q_1 < \infty$ we obtain essentially the same system of equations as before. One class of nonlinearities when this occurs is a set that may be called polynomial nonlinearities with

$$\sigma(z) = \sum_{N_1}^{N_2} c_n z^n,\ 0 < N_1 \leq N_2 < \infty \tag{38}$$

For example, if we assume that all trajectories that we wish to store are bounded, in order to introduce a sigmoid nonlinearity that is frequently used in memory models, one can take a cubic nonlinearity with

$$\sigma(z) = -(1/3)\,\alpha^2 z^3 + \beta^2 z,\ \alpha, \beta > 0 \tag{39}$$

and load only such trajectories for which $\sigma(z) \leq (2/3)\left(\beta^3/\alpha\right)$. Substitution of (39) into (35) results in

$$\left(i\rho_n b_n - \sum_{k=0}^{L} A_k \cdot \exp\left(-i\rho_n \tau_k\right) \cdot \left(\beta^2 b_n + \frac{1}{3}\sum_{p,q=-Q}^{Q} b_p b_q b_{n-p-q}\right)\right) = 0, \tag{40}$$

$$n = -Q_1, ..., Q_1$$

As in the linear case this equation can be still considered as an eigenvalue problem for b_n, except for a nonlinear operator $\tilde{A} = g(A)$. On the other hand this equation can also be considered as a linear equation on A_k for a given set of b_n. One novel feature that appears in the nonlinear case is that since we have a cubic nonlinearity, in general, up to three different values for each b_n are possible so that the term $\left(\beta^2 b_n + \frac{1}{3}\sum_{p,q=-Q}^{Q} b_p b_q b_{n-p-q}\right)$ has the same value and hence the solution for A_k is the same. This observation can be used for multiple memory storage of a set of M desired memories $\left\{b_n^B\right\}, B = 1, ..., M$. Consider the following system of equations

$$\left(\beta^2 b_n + \frac{1}{3}\sum_{p,q=-Q}^{Q} b_p b_q b_{n-p-q}\right) = C_n,\ n = -Q_1, ..., Q_1$$

This cubic system of $2Q+1$ equations has 3^{2Q+1} choices of solutions. As a result, in general, one can load 3^{2Q+1} memories corresponding to the same choice of matrices A_k. The same argument applies to any polynomial nonlinearity defined by (38) with the corresponding maximum number of memories growing as N_2^{2Q+1}. Of course for this storage method to be practical a prescription should be given for a way to store a given set of memories for a given nonlinearity, rather then choosing a nonlinearity to fit the needed set of memories as we described

above. In addition a thorough investigation of the basins of attraction of the multiple memories needs to be carried out. These and other questions shall be addressed in a future publication.

A problem can arise if Q_1 is sufficiently large, that is the system can become overdetermined if Q_1 is large enough. If Q_1 is finite the problem can be cured by choosing appropriately large L. The real problem appears when nonlinearity is such that $Q_1 = \infty$. Then, one would expect that the system (35) generically has no solutions. When $Q_1 = \infty$ our approach can be nevertheless applied approximately in the following sense. If $\Lambda_n(\tau_k; \{b_m\}) \to 0$ sufficiently fast as $n \to \infty$, we can truncate the system at a finite $Q_{\max}$ and the proceed as before with computing the A_k. Of course since we zeroed the coefficients $\Lambda_n(\tau_k; \{b_m\})$ for $n > Q_{\max}$ in effect we substituted the original nonlinearity $\sigma(z)$ with another one $\tilde{\sigma}(z)$ and the periodic extensions obtained for $\tilde{\sigma}(z)$ will not be periodic solutions for the original nonlinearity. However if the original nonlinear DDE is stable with regard to small perturbations of the solutions, the aperiodicity will not grow in time and will remain small. Therefore we still be able to store time-variable patterns, although with an additional error.

The criteria for the stability of the periodic extension follow from the stability analysis of the associated DDE. For the nonlinear case the stability of the nonlinear DDE is equivalent to the stability of the linearized version of the original DDE. The basic result about the stability of a linear DDE is that any solution of a DDE is stable if and only if the largest real part of the roots of the characteristic equation is non-positive. The criteria for the roots of the characteristic equation to lie in the complex left half-plane have been studied extensively. A summary of the results can be found in [9].

5 Parameter Adaptation Dynamics

If the construction of periodic extensions is realized in the physiology of the brain then there must exist an algorithm that adjusts weights and/or delays to arrive at the needed values so that a population of neurons learns the appropriate weights after many repetitions of exposure to the external input. In this section we present examples of such learning algorithms. Of course we do not claim that either of the algorithm is actually implemented in the brain. Neither is learning a central point of this paper. Nevertheless it is instructive to consider some possibilities to estimate the difficulty of the problem.

We shall consider that the delays are known and fixed and only the weight matrices A_k need to be determined. The delays can also be considered as variables instead or together with A_k . Investigation of this possibility we leave for elsewhere.

A learning algorithm is not difficult to construct by introducing a Liapunov dynamics in the space of weights so that at the fixed point of the evolution we obtain the needed periodic extension. The weight dynamics can be consequently constructed by defining an error functional that measures the distance from the desired solution to the initial condition and using it as a Liapunov function.

Consider, for example, "slow" learning done during the multiple presentations - "epochs" - of the same initial condition. The "m"th update of A_k would then be done once per "epoch": one update for each time interval $[m\ \tau_L, (m+1)\ \tau_L]$. As an example we can take the "m"th epoch error functional as

$$V_m[A] = \int_{m\ \tau_L}^{(m+1)\tau_L} dt\ \left|x(t) - \phi_0(t)\right|^2, \tag{41}$$

where $x(t)$ is the solution of the DDE with a given initial condition $\phi_0(t)$. "Fast" adaptation within the single epoch can also be defined in a similar way, although from the point of view of biology it might be more suitable for adaptation of delays. The error functional (41) is phenomenologically appealing since both $x(t)$ and $\phi(t)$ can be physically "measured" by the encoding population of neurons. The discrete evolution in the space of weights with fixed delays can then be written as

$$(A_k)_{m+1} = (A_k)_{m+1} - \varepsilon \frac{\partial}{\partial A_k} V_m[A], \tag{42}$$

where ε is a small parameter. If the fundamental solution can be computed then $x(t)$ in (41) is given by

$$x(t) = X(t-\tau_L)\,\phi(\tau_L) + \sum_{k=1}^{L} \int_{\tau_L-\tau_k}^{\tau_L} d\tau\ X(t-\tau-\tau_k)\ A_k\ \phi(\tau),$$

where the dependence of the fundamental solution $X(t)$ on the weights is given by (7,6). Computation of the appropriate derivatives of $V[A]$ involves computing the $A-$derivatives of the fundamental solution. These can be written out as

$$\frac{\partial X_{pq}}{\partial (A_k)_{ij}} = \frac{1}{2\pi i} \int_{(c)} d\lambda\ \exp\lambda(t-\tau_k) \left(H^{-2}(\lambda)\right)_{pi}\ \delta_{qj}, \tag{43}$$

which enables us to compute $\frac{\partial}{\partial A_k} V_m[A]$.

A more direct approach to learning the weights is to use the characteristic equation itself. We know from the discussion in the preceding sections that matrices A_k must be chosen in such a way that the characteristic equation has purely imaginary roots located at $\lambda_n = \frac{2\pi i}{\tau_L} n,\ n = -Q, ..., Q$. Therefore an alternative Liapunov function for the weight dynamics can be chosen as

$$V_H[A] = \frac{1}{2} \sum_{n=-Q}^{Q} \left|\det H(\lambda_n)\right|^2 \tag{44}$$

$$H(\lambda) = \lambda I - \sum_{k=0}^{L} A_k \exp(-\lambda\tau_k)$$

The weight dynamics induced by this Liapunov function is given by

$$\dot{A}_k = -\varepsilon\,\mathrm{Re}\left(\sum_{n=-Q}^{Q} \exp(-\lambda_n\tau_k) \det \bar{H}(\lambda_n)\, H^{-T}(\lambda_n)\right) \tag{45}$$

where bar denotes complex conjugation. This system of equations conceptually simpler than the one obtained from (41), since it does not involve integration over time.

Yet another approach is to use the explicit form of the solutions for A_k and define the norm in the space of matrices by

$$V_Y[A] = \frac{1}{2}\sum_{k=0}^{L} tr\left(A_k - Y_k\right)\left(A_k - Y_k\right)^T \tag{46}$$

$$Y_k = \sum_{k=0}^{2Q} T_{kn}^{-1}\Gamma_n$$

where T, Γ are defined by $(2829, 32, 33)$. Differentiation with respect to unknown A_k results in

$$\dot{A}_k = -\varepsilon\left(\left(A_k - Y_k\right)\right)$$

The three algorithms we presented all apply only to the linear case. For applications in biology the DDE are typically nonlinear. In this case one can use a stochastic annealing approach to the error functional $V_m[A]$ defined by (41). Another alternative is to restrict learning to the linear regime of the nonlinear DDE. This is possible because the typical nonlinearities in memory models, e.g. the Hopfield model, do have a region of definition of the nonlinearity where $\sigma(z) \approx z$.

6 Summary

In this paper we presented a method for approximate storage of the trajectories of nonlinear dynamical systems as periodic or almost periodic solutions of the linear and nonlinear delay-differential equations.

Although the main motivation for this work was to develop a model for time-variable pattern storage by the brain, the results might have wider applicability. Indeed, they indicate that linear delay differential equations are, in a sense, universal models for nonlinear dynamical systems. Choosing appropriately large N one in principle can store arbitrary (but finite) number of periodic orbits of a nonlinear dynamical system.

In addition to providing a method for storage of time-dependent patterns our model has other features that are attractive for a biological interpretation. For example, the method requires global synchronization of populations of neurons, since the initial conditions have to be represented via Fourier series with the same fundamental frequency $\frac{2\pi}{\tau_L}$. The maximum delay τ_L can be thought of the attention span of the population: periodic trajectories with period larger then τ_L cannot be stored. The model does not have unknown parameters. Once the delays are known the weights can be determined uniquely for a given trajectory of the environment.

There are a number of open questions about our method. Although the situation with the linear DDE seems to be clear, a more detailed investigation of the effect of the presence of nonlinearity should be carried out. One beneficial effect of nonlinearity could be multistability for a chosen set of the coefficients A_k, τ_k: the existence of different, initial condition dependent periodic solutions similar to the existence of multiple fixed points in the Hopfield model. This would make multiple trajectory storage more efficient: a linear DDE can store additional orbits only by increasing its dimension linearly with the number of the orbits.

We have shown that nonlinearity brings along one interesting feature: multistability. More than one periodic solution can be stored for a given set of matrices A_k. When nonlinearity is polynomial, one can estimate that maximum possible number of memories grows exponentially in the number of Fourier coefficients. Additional indication that the multistability should exist comes from considering a continuum limit of (35), a limit that is reasonable to take when there are large number of delays that depend smoothly on their index. This is the situation when neurons in the brain have large dendritic trees. With obvious definitions the continuum limit can be formally written as

$$\dot{\phi}(t) = \lambda \int_0^1 ds\ A(s) \cdot \sigma(\phi(t - \tau(s))) \tag{47}$$

for λ a parameter and for some monotonic delay function $\tau(s)$. Here we normalized the matrix-valued function $A(s)$ so that $tr\left(A(s)^T A(s)\right) = 1$. With the use of the Fourier transform this equation can be related to the Volterra and Hammerstein classes of nonlinear integral equations, which exhibit bifurcations in the parameter λ. The theory of nonlinear integral equations has been exhaustively studied and a number of criteria are available for determining the bifurcation points [10]. These depend mainly on the growth properties of the nonlinearity $\sigma(z)$. A more detailed analysis of (47) shall be considered elsewhere.

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References

[1] A. N. Jourjine, Factorization in Nonlinear Dynamical Systems, MPI-PKS preprint, in preparation.

[2] J. J. Hopfield, Neurons with Graded Response have Collective Computational Properties like Two-state Neurons,, Proc. Natl. Acad. Sci. USA, Vol. 81, pp. 3088-3092, 1984.

[3] A. N. Jourjine, A Liapunov Function for Additive Neural Networks and Nonlinear Integral Equations of Hammerstein Type, in Proc. Neural Networks

for Signal Processing III, C. A. Kamm, G. M. Kuhn, B. Yoon, R. Chelappa, S. Y. Kung Eds, IEEE, New York, 1993

[4] R. J. McEliece, E. C. Posner, E. R. Rodemich and S. S. Venkatesh, The Capacity of the Hopfield Associative Memory, IEEE Trans. on Information Theory, Vol. IT-33, pp.461-482, 1987.

[5] W. Ma, T. Hara, Y. Takeuichi, Stability of a 2-Dimensional Neural Network with Time Delays, Journal of Biological Systems, Vol. 8, pp. 177-193, 2000.

[6] J. Foss, A. Longin, B. Mensour, and J. Milton, Multistability in Delayed Recurrent Loops, Phys. Rev. Lett. Vol.74, pp708-711, 1996.

[7] H. C. Tuckwell, Introduction to Theoretical Neurobiology, Cambridge University Press, 1988.

[8] R. Bellman and K. L. Cooke, Differential Difference Equations, Academic Press, 1963.

[9] J. K. Hale and S. M. V. Lunel, Introduction to Functional Differential Equations, Springer Verlag, New York, 1993.

[10] K. Yosida, Lectures on Differential and Integral Equations, Dover Publications, New York, 1991.